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Unsteady-state transfer between a sphere and a surrounding stationary medium with application to arrays of spheres

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Abstract—A series solution is presented for transfer between a sphere and a surrounding stationary medium enclosed in a concentric insulating sphere. It is shown that after a period, short compared to the time necessary to transfer a small proportion of the maximum quantity transferable, the first term of the series dominates. The transfer process then proceeds with a constant Nusselt or Sherwood number determined only by the ratio of the radii of the spheres. For large ratios $Nu = Sh = 2$. For a ratio of 5 : 1, the number is about 3 and for lower ratios it increases rapidly. For an array of similar spheres enclosed in an insulated volume, it is possible to identify an envelope around each sphere across which there is no flux. By approximating each envelope to a sphere, the analysis becomes applicable to such problems as evaporation from a cloud of droplets.

1. INTRODUCTION

HEAT and mass transfer from spheres in a stationary atmosphere is of practical interest for two reasons. First, for small unsupported droplets, as may be found in spray dryers, combustion systems etc., the relative velocity between gas and droplet is small for the majority of time taken for the droplet to evaporate. Secondly, in systems in which the relative velocities are not negligible, the asymptotic behaviour of the transfer equations as the relative velocity approaches zero must correspond to the behaviour in a stationary medium [1].

Relationships proposed to date take steady-state models. The most widely used, namely $Nu = Sh = 2$ considers steady transfer from an isolated sphere to an infinitely distant sink. Alternative relationships have been given by Cornish [2] who treats the case of steady transfer from a finite array of spheres in an infinite medium to an infinitely distant sink, and by Friedlander [3] who quotes the case of steady transfer from a sphere to a surrounding concentric spherical sink at a finite distance. In all these cases the driving force for transfer is taken to be the temperature (or, for mass transfer, concentration) difference between the sphere and the sink.

The motivation for the work in this paper is the observation that in most cases of practical interest the objective of the process is to transfer heat or material between the sphere and the surrounding medium. Thus, as a result of the transfer process, we expect the medium to become for example, hotter or more humid. In these circumstances, the medium clearly acts as a sink. This position is recognized in those calculations in which the driving force for transfer is taken as the difference between sphere surface temperature and mean bulk temperature of the fluid. It is the objective of this work to derive theoretical values for Nusselt and Sherwood numbers applicable to transfer between spheres and a surrounding medium for which mean bulk conditions can be used for calculation of driving force. In the absence of an external sink, this situation can only be modelled by unsteady (transient) equations.

In an array of similar spheres, all transferring heat or material to the same medium, it is possible to identify an envelope around each sphere across which there is no flux. The transfer from each sphere is then identical to that which would occur if a perfectly insulating surface were placed around the relevant envelope. In this way transfer within an array of spheres can be studied by considering transfer from one single sphere placed in a medium contained in a finite insulated envelope. For an indefinitely large cubic arrangement of spheres with an initially uniform temperature and concentration in the medium, it is immediately apparent from symmetry that the envelopes are cubic and each envelope is concentric with its enclosed sphere. For a random arrangement of spheres the envelopes will be irregular (and may change with time). The general case is difficult to solve but in this paper we make the simplifying assumption that each envelope can be replaced by a spherical envelope of equal volume and that the resulting envelope is concentric with its enclosed sphere. All previous analyses for transfer from individual spheres in an array have also assumed spherical symmetry (e.g. the suggestion of Zabrodsky [4] is very similar). The analysis of this paper is, in this way, comparable to previous analyses. It is capable of giving a direct indication of the consequences of a transient solution (not introducing an artificial sink)

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NOMENCLATURE

A	coefficient in equation (2)
B_i	coefficient in equation (2)
C	heat capacity of medium
d	diameter of hot sphere, $2r_1$
F_i	see equation (A1)
f	Q_{t1}/Q_{01}
h_0	heat transfer coefficient based on a ΔT_0, i.e. $k(\partial T/\partial r)_{r=r_1}/\Delta T_0$
h_m	heat transfer coefficient based on ΔT_m, i.e. $k(\partial T/\partial r)_{r=r_1}/\Delta T_m$
k	thermal conductivity of medium
M_i	see equation (A3)
N_i	see equation (A3)
Nu_f	$2/(1-r_1/r_2)$, Nusselt number for finite sink model
Nu_m	Nusselt number based on bulk–mean temperature difference, $h_m d/k$
Nu_0	Nusselt number based on overall temperature difference, $h_0 d/k$
n	see equation (A6)
P_i	coefficient in equation (2)
Q	total rate of heat transfer from sphere at time t
Q_i	coefficient in equation (2)
Q_{01}	initial rate of heat transfer from sphere due to 1st term of equation (2)
Q_{t1}	rate of heat transfer from sphere due to 1st term of equation (2) after time t
R	$1-r_1/r_2$
r	radial distance from centre of hot sphere
r_1	radius of hot sphere
r_2	radial distance to insulating envelope
S	surface area of sphere, $4\pi r_1^2$
s_i	parameters in equation (2)
T	temperature
T_1	temperature at $r = r_1$
T_2	temperature at $r = r_2$
T_m	mean bulk temperature of medium
ΔT_0	$T_2 - T_1$
ΔT_m	$T_m - T_1$
t	time.

Greek symbols

α	$k/(C\rho)$
δ	$r_2 - r_1$, on separation of hot surface from insulated surface
ε	voidage of array of spheres (see equation 13)
ρ	density of medium
τ	time taken for 1st term heat transfer rate to drop to Q_{t1}, divided by time taken for 99% of heat to be transferred (note τ would be doubled, if we took time for only 90% of heat to be transferred)
θ_i	$\omega_i r_2$
ω_i	coefficient in equation (2).

Subscripts

i	term number, can take values 1, 2, 3, ...

without introducing complications of geometry differing from previous analyses. The maintenance of spherical symmetry also, of course, allows the simplest mathematical treatment.

Introducing a transient treatment in principle invalidates the use of a Nusselt or Sherwood number to describe the heat or mass transfer process because the transfer rate will depend on the initial conditions. Thus, if a hot sphere is introduced instantaneously into a cold fluid the initial temperature gradient at the surface is infinite and the heat transfer rate from the sphere very high, independent of the overall temperature difference 'driving force'. Conversely, if a hot sphere surrounded by a fluid layer of its own temperature is introduced into a cold fluid, the initial heat transfer rate from the sphere will be zero, independent of the overall temperature difference.

This theoretical difficulty is reflected in practice. Thus for real dispersed systems it is impossible to define the initial temperature concentration gradients as, for example, a droplet is projected from an atomiser. Although not treated in this paper, a similar difficulty applies to momentum transfer with ill-defined initial velocity profiles. No treatment has previously been given to indicate the longer term effects of these initial differences in temperature and concentration profiles.

In this paper we show that the effect of initial profiles is short-lived and give a treatment to show how the concept of 'short' may be quantified for practical application. After this initial short period calculations based on Nusselt and Sherwood numbers are valid and the paper gives values for Nu and Sh as a function of the void fraction of the array which shows that transfer rates should be higher than those based on $Nu = Sh = 2$. The theoretical results are thus consistent with the experimental results of workers such as Rowe and Claxton [1] (although since they did not surround their target sphere with similar spheres, the results will not be directly comparable).

2. THEORY

For simplicity of presentation, the treatment refers to a hot sphere in a cool medium but, with straightforward changes, it is equally applicable to heat or mass transfer to or from a sphere.

The single-sphere model

The model system taken is a hot sphere, held at constant temperature, surrounded by a cool stationary medium of constant density, heat capacity and conductivity which is enclosed in a concentric insulating sphere.

The equation for heat transfer through the medium (as given in any standard text on unsteady-state heat transfer) is:

$$\frac{1}{r}\frac{\partial^2(rT)}{\partial r^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

where:

$$\alpha = k/(C\rho)$$

(symbols as defined in the Nomenclature).

A general solution of equation (1) is:

$$T = A + \frac{1}{r}\left[\sum P_i\, e^{-\alpha\omega_i^2 t} \sin \omega_i r + \sum Q_i\, e^{-\alpha\omega_i^2 t} \cos \omega_i r\right]$$

which can be written

$$T = A + \frac{1}{r}\sum B_i\, e^{-\alpha\omega_i^2 t} \sin \omega_i(r+s_i). \tag{2}$$

The appropriate boundary conditions are:

(I) At $r = r_1$, T = constant, all t. Since all terms except A may vary with t, we have that the coefficient of $\exp\{-\alpha\omega_i^2 t\}$ must be zero for all i when $r = r_1$. It follows, since B_i may be fixed to match any arbitrary initial temperature profile, that

$$\sin \omega_i(r_1+s_i) = 0$$

or

$$s_i = -r_1, \text{ all } i. \tag{3}$$

(Adding constant multipliers of π does not give rise to independent terms.)

Equation (2) then becomes

$$T = A + \frac{1}{r}\sum B_i\, e^{-\alpha\omega_i^2 t} \sin \omega_i(r-r_1). \tag{4}$$

(II) At $r = r_2$, $\partial T/\partial r = 0$, all t. As for condition (I), each term must be zero, from which it follows that

$$\sin \omega_i(r_2-r_1) = r_2\omega_i \cos \omega_i(r_2-r_1) \tag{5}$$

or

$$\tan(\theta_i R) = \theta_i. \tag{6}$$

Note $0 < R < 1$, so that there is always a solution θ_1 such that $0 < \theta_1 R < \pi/2$.

The other solutions are in the ranges

$$\pi < \theta_2 R < \pi + \pi/2$$

$$2\pi < \theta_3 R < 2\pi + \pi/2 \text{ etc.,}$$

and are readily obtained numerically.

Nusselt number. It is shown in Appendix 1 that, after a short period, the second and subsequent terms of the summation in equation (2) are negligible compared to the first. Hence, to good approximation

$$T = A + \frac{B_1}{r}\, e^{-\alpha\omega_1^2 t} \sin \omega_1(r-r_1). \tag{7}$$

Equation (7) indicates that the shape of the temperature profile remains constant as t varies, from which it follows that the heat transfer rate can be computed from a suitable Nusselt number.

The appropriate Nusselt number, $Nu = hd/k$, can be obtained from the two relationships for heat transfer rate from the sphere, one based on heat transfer coefficient, h, the other on boundary temperature gradient, namely:

$$Q = hS\Delta T = kS(\partial T/\partial r)_{r=r_1}$$

giving

$$Nu = 2r_1(\partial T/\partial r)_{r=r_1}/\Delta T \tag{8}$$

$(\partial T/\partial r)_{r=r_1}$ is obtained from equation (7)

$$\left.\frac{\partial T}{\partial r}\right|_{r=r_1} = \frac{B_1\omega_1\, e^{-\alpha\omega_1^2 t}}{r_1}.$$

From which:

$$Nu = 2B_1\omega_1 \exp\{-\alpha\omega_1^2 t\}/\Delta T. \tag{9}$$

Previous, steady-state, analyses have based Nu on overall temperature difference, in this case $(T_2 - T_1)$.

$$T_2 - T_1 = B_1 \exp\{-\alpha\omega_1^2 t\} \sin \omega_1(r_2-r_1)/r_2$$

so that based on overall temperature difference

$$Nu_0 = 2\omega_1 r_2/\sin \omega_1(r_2-r_1)$$

or, substituting for $\sin \omega_1(r_2-r_1)$ from equation (5)

$$Nu_0 = 2\sqrt{1+\theta_1^2}. \tag{10}$$

It is the objective of this paper to obtain a Nusselt number based on mean bulk temperature which is easily obtained by overall heat balance. (The temperature at the envelope, T_2, is not directly obtainable from practicable measurements.)

The mean bulk temperature is given by:

$$T_m = \int_{r_1}^{r_2} Tr^2\, dr \bigg/ \int_{r_1}^{r_2} r^2\, dr.$$

Substituting for T from equation (7) and integrating by parts for the numerator, gives after cancelling terms shown to be equal in equation (5)

$$T_m - T_1 = \frac{3B_1 r_1 \exp\{-\alpha\omega_1^2 t\}}{\omega_1(r_2^3 - r_1^3)}. \tag{11}$$

Combining equations (9) and (11), we get

$$Nu_m = \frac{2\theta_1^2[1-(r_1/r_2)^3]}{3(r_1/r_2)}. \tag{12}$$

Application to arrays of spheres. In considering heat transfer from an array of spheres, we have introduced the concept that each sphere can be treated as if surrounded by an insulating envelope. The radius r_2 is

Table 1. Dependence of Nu_m on ε and comparison with finite sink values

ε	r_1/r_2	Nu_m	Nu_f
1.0	0.0	2.0	2.0
0.999999	0.01	2.0365	2.0202
0.999992	0.02	2.0742	2.0408
0.999973	0.03	2.1129	2.0619
0.999875	0.05	2.1938	2.1053
0.999	0.1	2.4179	2.2222
0.992	0.2	2.9788	2.5000
0.973	0.3	3.7413	2.8571
0.936	0.4	4.8030	3.3333
0.875	0.5	6.3398	4.0000
0.784	0.6	8.7041	5.0000
0.657	0.7	12.718	6.6667
0.488	0.8	20.849	10.0000

then the radius of a sphere of volume equal to that of the envelope. The void fraction of one envelope containing a hot sphere is equal to the void fraction of the whole array of spheres [15], and is given by:

$$\varepsilon = 1-(r_1/r_2)^3$$

or

$$(r_1/r_2) = (1-\varepsilon)^{1/3}. \quad (13)$$

Computed results are presented in the next section giving Nu as a function of ε, which is directly measurable whereas r_2 is not.

3. COMPARISON WITH STEADY-STATE ANALYSES

Table 1 gives values of Nu_m calculated as a function of ε, from equations (13), (6) and (12).

It is seen that even for fairly low sphere concentrations, the deviation of Nu_m from the 'classical' value of 2.0 is appreciable.

Given the void fraction of the system, it is straightforward to obtain the appropriate value of Nu_m from Table 1 which can then be used in conventional calculations where $Nu = 2$ has been used in the past.

Three further comparisons with steady-state analyses can be made, (a) the temperature profiles from which the Nusselt numbers are derived, (b) an analytical approximation to Nu for small values of (r_1/r_2), and (c) an analytical approximation to Nu for large values of (r_1/r_2).

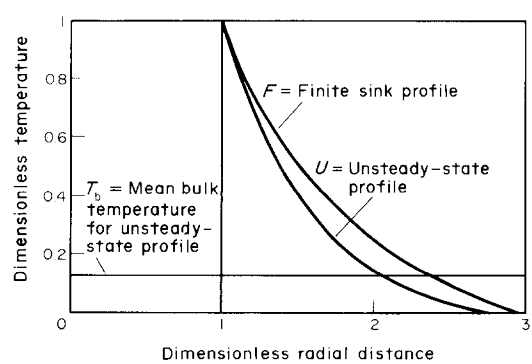

Fig. 1.

(a) *Temperature profile*

The unsteady state profile is compared with the finite state temperature profile in Fig. 1 for $r_2/r_1 = 3$. Both profiles are plotted in dimensionless form when equation (7) becomes:

$$\frac{T-T_2}{T_1-T_2} = 1-\frac{r_2}{r}\frac{\sin\left[\theta_1(r/r_2-r_1/r_2)\right]}{\sin\left[\theta_1(1-r_1/r_2)\right]}$$

and the corresponding finite sink profile is:

$$\frac{T-T_2}{T_1-T_2} = \frac{r_1/r-r_1/r_2}{1-r_1/r_2}.$$

The figure also shows the mean bulk temperature for the transient profile, which, from equations (7) and (11) is given by:

$$\frac{T_m-T_1}{T_2-T_1} = \frac{3(r_1/r_2)}{\theta_1[1-(r_1/r_2)^3]\sin\theta_1 R}.$$

In this case the transient Nusselt number is about 35% greater than the finite sink number. Just under half the difference is attributable to taking the mean bulk temperature rather than the overall temperature difference and just over half the difference is attributable to the steeper transient profile near to the surface of the hot sphere.

(b) *Nusselt number for $r_1 \ll r_2$*

For $r_1 \ll r_2$, equation (6) becomes

$$\theta_1 R = \tan^{-1}\theta_1$$

$$\simeq \theta_1-\frac{\theta_1^3}{3}+\frac{\theta_1^5}{5}\cdots,$$

i.e.

$$\theta_1^2 \simeq 3(1-R)+\frac{27}{5}(1-R)^2$$

$$= 3\left(\frac{r_1}{r_2}\right)+\frac{27}{5}\left(\frac{r_1}{r_2}\right)^2 \quad (14)$$

giving

$$Nu_0 \simeq 2+3\left(\frac{r_1}{r_2}\right) \quad (15)$$

and

$$Nu_m \simeq 2+\frac{18}{5}\left(\frac{r_1}{r_2}\right). \quad (16)$$

For comparison, expressed in the same form, the finite sink equation is:

$$Nu_f \simeq 2+2\left(\frac{r_1}{r_2}\right)+\cdots \quad (17)$$

Overall, it is seen that Nu_m differs from 2.0 by 80% more than Nu_f, 50% as a consequence of the steeper transient profiles [equation (15)] and a further 30% as a consequence of taking the mean bulk temperature instead of the overall temperature difference.

(*c*) *Nusselt number for* $r_1 \simeq r_2$

When the enveloping sphere is close to the hot sphere, $r_1 \simeq r_2$ and $R \ll 1$. Equation (6) can then only be solved for very large values of θ_1, so that

$$\theta_1 R = \tan^{-1}\theta_1 \simeq \pi/2$$

and

$$\theta_1 \simeq \tfrac{1}{2}\pi/R. \quad (18)$$

Substituting equation (18) into equation (12) gives:

$$Nu_m \simeq \frac{\pi^2}{2[1-(r_1/r_2)]}. \quad (19)$$

The corresponding equations for Nu_0 and Nu_f are:

$$Nu_0 = \pi/[1-(r_1/r_2)]$$

and

$$Nu_f = 2/[1-(r_1/r_2)].$$

Nu_m is seen to be $\pi^2/4$ times greater than Nu_f, of which factor $\pi/2$ is a consequence of the steeper temperature gradient and a further factor $\pi/2$ a consequence of taking the mean bulk temperature driving force.

4. TRANSFER TO CLOSELY-SPACED SPHERES

The approximation that a sphere within an array of spheres can be treated as a single sphere within an insulated sphere, is less tenable with closely spaced spheres than with widely-separated spheres. The model may nevertheless give useful qualitative information. As the concentric spheres become close, the model approaches a system of heat transfer to a medium between two parallel plates, one heated the other insulated. The only characteristic distance is the plate separation and it would be expected that a Nusselt number based on plate separation would be appropriate. The appropriate number is obtained by putting $\delta = r_2 - r_1$, giving:

$$Nu_\delta = \frac{h\delta}{k} = \frac{1}{2}\frac{Nu_m}{r_1}(r_2 - r_1).$$

When $r_2 \simeq r_1$ equation (19) is applicable so that

$$Nu_\delta \simeq \frac{\pi^2 r_2}{4r_1} \simeq \frac{\pi^2}{4}$$

which is independent of sphere radius.

With such close spacing, the assumption that transfer rate is independent of initial temperature profile also becomes less tenable (see Appendix 1). The analysis nevertheless leads to the conclusion that transfer to a stationary fluid from closely spaced particles immersed in the fluid may be calculable using a Nusselt number based on average separation distance. This number may be a universal constant of order of magnitude $2Nu_\delta$ or about 5.

5. CONCLUSIONS

(i) It has been shown to be practicable to compute heat or mass transfer from a sphere to a surrounding medium using a transient analysis which indicates that a Nusselt or Sherwood number based only on the geometry may be applicable. This situation arises frequently in practice and no previous comparable theoretical analysis has been given.

(ii) For a sphere surrounded by a stationary medium enclosed in an insulated sphere the Nusselt (or Sherwood) number depends on the ratio of sphere diameters. The Nusselt number is significantly higher than for transfer to an infinitely distant sink. Thus, for a ratio of sphere diameters of 5:1 (when the hot sphere takes up less than 1% of the volume of the medium), the Nusselt number is 3, which is significantly above the infinite sink value of 2 and the finite sink value of 2.5. For smaller sphere diameter ratios the differences become much greater.

(iii) It is argued that the results are applicable to arrays of spheres and appropriate formulae are given based on voidage.

(iv) An indicative analysis is given suggesting that for closely packed particles a universal Nusselt number based on average particle separation distance may be applicable.

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APPENDIX 1: RELATIVE MAGNITUDE OF TERMS IN SERIES SOLUTION

Equations (4) and (6) give a general solution to the problem treated. In general, we have no information on the values of B_1, B_2... applicable to practical problems. It is, however, possible (using an analysis similar to that for obtaining coefficients in a Fourier series) to find values for particular hypothetical profiles. For an initially uniform profile the coefficients are of similar order to magnitude. For monotonically decreasing profiles, the coefficients show a tendency to decrease. Intuitively these situations are more likely to arise in practice, on the grounds that some transfer is likely during the formation or placing of the spheres.

Table A1. Values for N_1 for a range of f and ε

r_1/r_2	ε	$f = 0.9$ $\tau = 0.023$	0.8 0.048	0.5 0.15	0.2 0.35
0.01	0.999999	3.1×10^{-30}	—	—	—
0.02	0.999992	6.0×10^{-15}	2.9×10^{-32}	—	—
0.03	0.999973	6.9×10^{-10}	1.9×10^{-21}	—	—
0.05	0.999875	7.0×10^{-6}	7.7×10^{-15}	—	—
0.1	0.999	0.0061	1.9×10^{-6}	2.0×10^{-20}	—
0.2	0.992	0.15	0.0025	2.0×10^{-10}	2.9×10^{-24}
0.3	0.973	—	0.026	4.0×10^{-7}	1.7×10^{-16}
0.4	0.936		0.079	1.7×10^{-5}	1.2×10^{-12}
0.5	0.875	—	0.15	1.6×10^{-4}	2.4×10^{-10}
0.6	0.784	—	—	6.8×10^{-4}	7.9×10^{-9}
0.7	0.657	—	—	0.0019	9.5×10^{-8}
0.8	0.488	—	—	0.0041	6.0×10^{-7}
1.0	0	—	—	0.012	7.7×10^{-6}

This analysis is made on the assumption that the coefficients B_i are initially of similar magnitude. In order to trace the variation of relative magnitude with time, we express equation (4) as:

$$T = A + \sum F_i \tag{A1}$$

where:

$$F_i = B_i \exp\{-\alpha\omega_i^2 t\} \sin \omega_i(r - r_1)/r.$$

In order to assess the importance of the terms, we determine their contribution to the heat transfer rate, Q, from the hot sphere:

$$Q \propto \left.\frac{\partial T}{\partial r}\right|_{r=r_1} = \sum \left.\frac{\partial F_i}{\partial r}\right|_{r=r_1} \tag{A2}$$

where from equation (A1):

$$\left.\frac{\partial F_i}{\partial r}\right|_{r=r_1} = B_i\omega_i \exp\{-\alpha\omega_i^2 t\}/r_1.$$

The relative importance of successive terms in determining heat transfer rate is given by:

$$M_i = \left.\frac{\partial F_{i+1}}{\partial r}\right|_{r=r_1} \div \left.\frac{\partial F_i}{\partial r}\right|_{r=r_1}$$

$$= \frac{B_{i+1}\omega_{i+1}}{B_i\omega_i} \frac{\exp\{-\alpha\omega_{i+1}^2 t\}}{\exp\{-\alpha\omega_i^2 t\}}$$

$$= \frac{B_{i+1}}{B_i} N_i. \tag{A3}$$

Where B_{i+1}/B_i is a ratio of order unity. It is, therefore, possible to ascertain an order-of-magnitude estimate of M_i by determining N_i.

All terms reduce with time and we will now show that as $\partial T_i/\partial r$ decreases N_i decreases rapidly.

Consider the time interval during which $\partial T_1/\partial r$ has decreased by a factor, f, then the heat transfer rate is given by:

$$Q/Q_{01} = B_1 f + B_2 N_1 f + B_3 N_2 N_1 f + \cdots$$

where Q_{01} is the initial heat transfer rate resulting from term 1.

We will perform the analysis only for N_1, a similar analysis applies for other terms.

From equation (A2)

$$f = \exp\{-\alpha\omega_1^2 t\}. \tag{A4}$$

From equation (A3)

$$N_1 = \exp\{-\alpha(\omega_2^2 - \omega_1^2)t\}\omega_2/\omega_1. \tag{A5}$$

Combining equations (A4) and (A5), we get

$$N_1 = f^n \omega_2/\omega_1$$

where:

$$n = (\omega_2/\omega_1)^2 - 1. \tag{A6}$$

Thus N_1 depends only on (ω_2/ω_1) which, in turn depends only on r_1/r_2 (or ε). Table A1 gives values of N_1 for a range of values of f and r_1/r_2. In order to emphasize the fact that the initial high heat transfer rates only last a short fraction of the time required to transfer an economic amount of heat, the table also includes values of τ, the ratio of the time required for the heat transfer rate to drop to f to the time required to transfer 99% of the total heat transferrable, i.e. $\tau = \ln(f)/\ln(0.01)$.

Note that the calculation for $r_1/r_2 = 1$ represents an asymptotic value at which:

$$\theta_1 R \to \pi/2$$

and

$$\theta_2 R \to 3\pi/2,$$

so that

$$\omega_2/\omega_1 \to 3$$

and

$$N_1 \to 3f^8.$$

Note also that, for small values of r_1/r_2 (i.e. cases in which it may have been considered valid to use the infinitely distant sink model giving $Nu = 2$), N_1 rapidly becomes vanishingly small. Thus for $r_1/r_2 = 0.01$, the second term contributes less than 1.0×10^{-29} (3.1×10^{-30} in table) for around 98% (1–0.023) of the time needed to heat the medium to 99% of its final temperature. This represents over 95% of the time ($1-2 \times 0.023$) needed to reach 90% of its final temperature. In these circumstances, the error in ignoring terms beyond F_1 is entirely negligible, i.e. beyond the precision of most computers and certainly beyond the precision of any physical data that might be used to compute Nu etc.

With larger values of r_1/r_2, the error becomes greater. However, even taking the extreme case of $r_1/r_2 = 1$, we find that for $f = 0.5$, $N_1 = 0.012$. Thus the error in ignoring terms beyond F_1 is less than about 1% for 85% (i.e. $1 - 0.15$) of the time needed to reach 99% of the final temperature, or 70% of the time needed to reach 90% of the final temperature. For 65% of the time, the error would be of order parts per million (7.7

$\times 10^{-6}$). The calculation of the heating time for a practical problem would thus be within a few percent ignoring terms beyond F_1; far more accurate that a calculation based on the steady-state model where the corresponding error in the calculated heat transfer rate approaches a factor of 5 [see equation (19)].

The table indicates clearly that, for most practical applications, the error will be far less than for this extreme case, so that a transient analysis based on taking equation (A1) only as far F_1 as is fully justified.

The numerical calculations in this appendix required the iterative solution of equation (6) which was facilitated by the good initial estimates given in Appendix 2.

APPENDIX 2: APPROXIMATE ANALYTICAL EXPRESSIONS

For approximate calculation, it may be noted that the two asymptotic solutions for θ_1 [equations (14) and (19)] can be smoothly joined to give:

$$\theta_1 \cong \frac{\sqrt{(r_1/r_2)}\,[\pi/2+(\sqrt{3}-\pi/2)(1-r_1/r_2)]}{(1-r_1/r_2)}$$

with a maximum error of approx. 0.2% for $0 < r_1/r_2 < 1$.

This approximate solution is also a useful starting estimate for computing θ_1 exactly. Similar simple expressions apply for θ_i, $i > 1$.

TRANSFERT VARIABLE ENTRE UNE SPHERE ET UN MILIEU ENVIRONNANT STATIONNAIRE, AVEC APPLICATION A UNE RANGEE DE SPHERES

Résumé—Une solution série est présentée pour le transfert entre une sphère et un milieu environnant stationnaire enfermé dans une sphère isolante concentrique. On montre qu'après une période courte comparée au temps nécessaire pour transférer une petite proportion du maximum transférable, le premier terme de la série domine. Le transfert est alors caractérisé par un nombre de Nusselt, ou un nombre de Sherwood, constant déterminé seulement par le rapport des rayons des sphères. Pour des grands rapports, $Nu = Sh = 2$. Pour un rapport 5 : 1, le nombre est proche de 3 et pour des rapports plus faibles, il croît rapidement. Pour une rangée de sphères similaires enfermées dans un volume isolé, il est possible d'identifier une enveloppe, autour de chaque sphère, à travers laquelle il n'y a pas de flux. En approchant chaque enveloppe par une sphère, l'analyse devient applicable à des problèmes comme l'évaporation à partir d'un nuage de gouttelettes.

INSTATIONÄRE ÜBERTRAGUNGSVORGÄNGE ZWISCHEN EINER KUGEL UND EINEM UMGEBENDEN, UNBEWEGTEN MEDIUM UND DIE ANWENDUNG AUF KUGELANORDNUNGEN

Zusammenfassung—Für die Übertragungsvorgänge zwischen einer Kugel und einem umgebenden, unbewegten Medium, eingeschlossen in eine konzentrische isolierte Kugel, wird eine Lösung in Form einer Reihe angegeben. Es wird gezeigt, daß bereits nach einer Zeit, welche gegenüber der zum Übertragen auch nur eines kleinen Anteils der maximal übertragbaren Menge notwendigen Zeit kurz ist, das erste Glied der Reihe dominiert. Der Übertragungsprozeß läuft dann mit einer konstanten Nusselt-oder Sherwood-Zahl ab, welche nur durch das Verhältnis der Kugelradien bestimmt wird. Für große Verhältnisse ist $Nu = Sh = 2$. Für ein Verhältnis von 5 : 1 sind die Kennzahlen ungefähr 3, und für niedrigere Verhältnisse steigen sie rasch an. Für eine Anordnung von ähnlichen Kugeln, welche in einem isolierten Volumen eingeschlossen sind, ist es möglich, um jede Kugel eine Hülle zu erkennen, an welcher kein Fluß mehr auftritt.

НЕСТАЦИОНАРНЫЙ ПЕРЕНОС МЕЖДУ СФЕРОЙ И ОКРУЖАЮЩЕЙ НЕПОДВИЖНОЙ СРЕДОЙ И ЕГО ПРИМЕНЕНИЕ ДЛЯ СЛУЧАЯ СИСТЕМЫ СФЕР

Аннотация—Представлено решение в виде ряда для процесса переноса между сферой и окружающей неподвижной средой, помещенной в концентрическую изолирующую сферу. Показано, что после периода, малого по сравнению со временем, необходимым для переноса небольшой части максимального количества переносимой величины, преобладает первый член ряда. Далее процесс переноса продолжается при постоянных числах Нуссельта или Шервуда, определяемых только отношением радиусов сфер. Для больших значений этого отношения получено $Nu = Sh = 2$. При величине отношения, равной 5:1, число Nu или Sh приблизительно равно 3, а для меньших значений отношения оно резко возрастает. Для системы подобных сфер, помещенных в изолированный объем, можно ввести воображаемую оболочку вокруг каждой сферы, на которой поток переносимой величины равен нулю. Если аппроксимировать каждую такую оболочку сферой, то проведенный анализ становится применимым к таким задачам как испарение в облаке капель.